On asymptotic behavior of portfolio risks and goodness of elliptical approximations

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Multivariate risks

- Risk factors that a portfolio has exposure to can be represented as a random vector

$$X = (X_1, \dots, X_d)'$$

- Given weights $\omega = (\omega_1, \dots, \omega_d)$, the total portfolio loss can be modeled as

$$L = \omega^T X = \sum_i \omega_i X_i$$

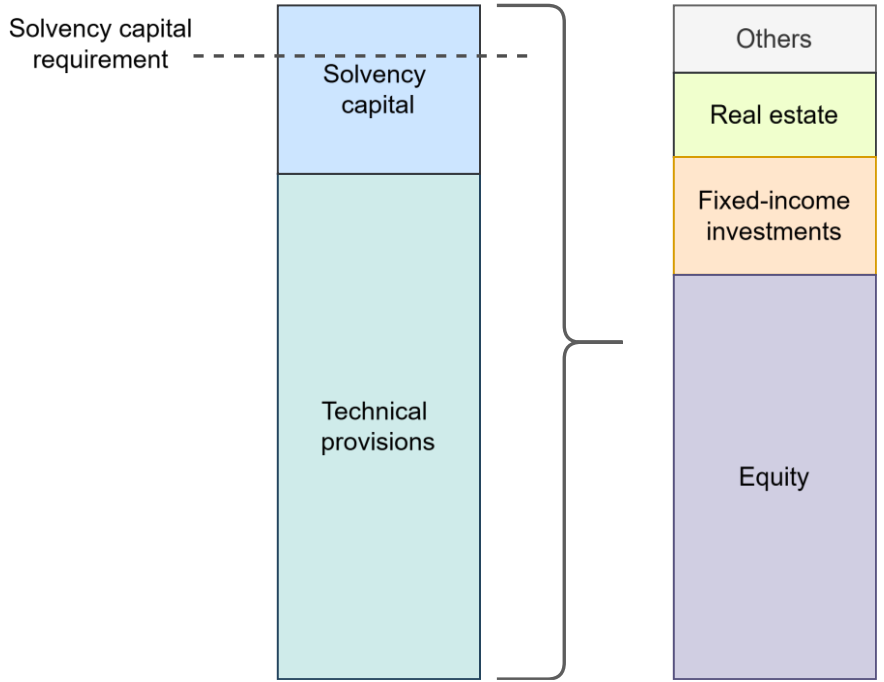
- To determine the risk, in regulatory context, we need to estimate

$$VaR_\alpha(\omega^T X) = \inf\{l \in \mathbb{R} : P(\omega^T X > l) \leq 1 - \alpha\}$$

Solvency capital requirement (SCR)

- The total value of assets should always exceed the value of liabilities
- To account for **fluctuations in asset values**, there is a regulatory safety margin – solvency capital requirement (SCR)
- Determined by a **similar mathematical formula** in many regulatory frameworks

Solvency capital requirement (SCR)



Elliptical approximations

Solvency II

$$\mathrm{SCR} = \sqrt{\sum_{i,j} Corr_{i,j} \times SCR_i \times SCR_j}$$

The Finnish statutory private sector pension scheme

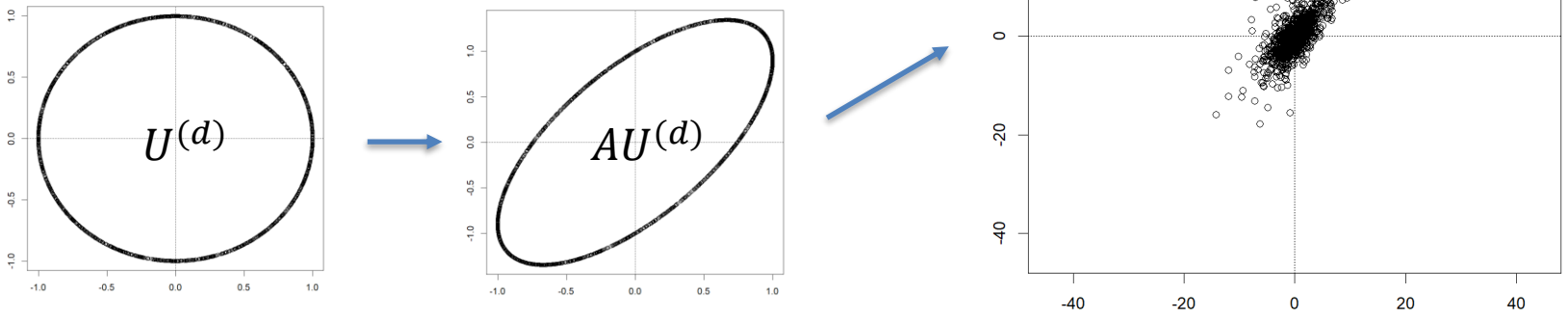
$$\mathrm{SCR} = -\sum_{i=1}^{18} \mu_i + \sqrt{\sum_{i=1}^{18}\sum_{j=1}^{18} \rho_{ij}(V_i - \mu_i)(V_j - \mu_j) + \sum_{i}^{4} \beta_i^2 B_i^2 + K}$$

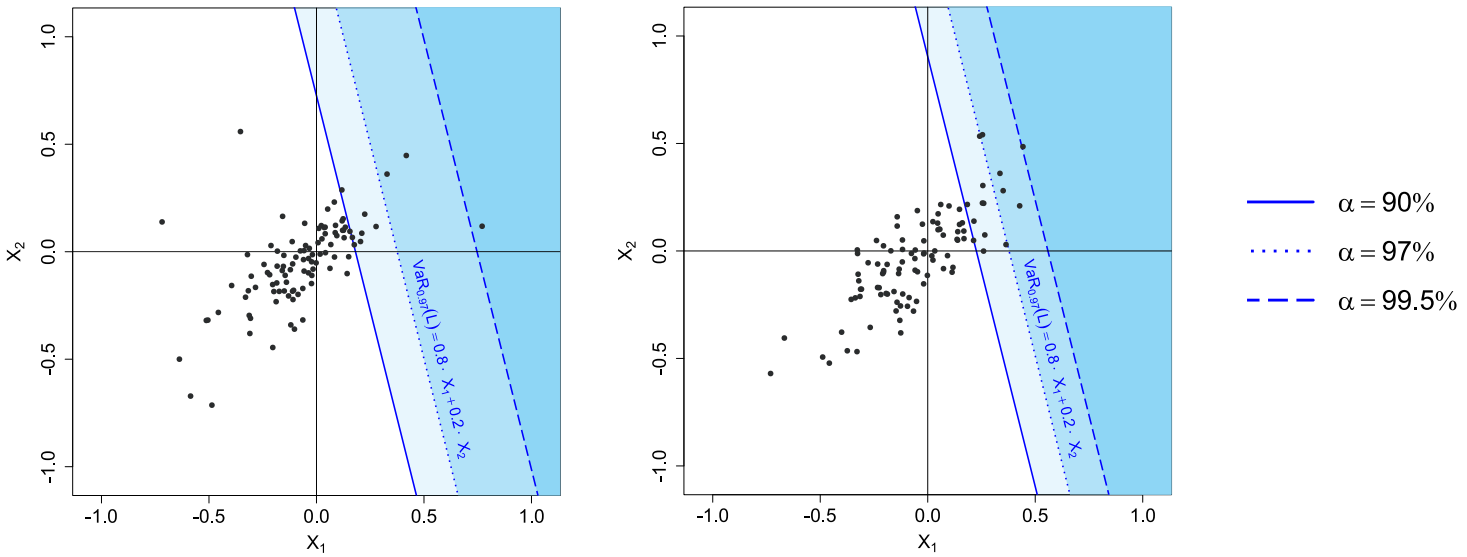
- **Elliptical distributions**

$$VaR_\alpha(\omega^T X) = \sum_i \omega_i E(X_i) + \sqrt{\sum_i \sum_j \omega_i \omega_j \rho_{ij} VaR_\alpha\big(X_i - E(X_i)\big) VaR_\alpha\Big(X_j - E\big(X_j\big)\Big)}$$

Elliptical distributions

$$X =^{D} \mu + R_d A U^{(d)}$$





R_d and ruin set $\{X \in \mathbb{R}^d : \omega^T X > VaR_\alpha(\omega^T X)\}$ bounds

***t*-distribution** with $\nu = 4$ (left) and **multivariate normal** (right), calibrated to match correlations and VaR 97 %

Elliptical distributions

Elliptical distributions

Asymptotic dichotomy of R_d

Regularly varying tail

$$\lim_{x\to\infty} \frac{P(R_d>ax)}{P(R_d>x)} = a^{-\nu},\ a>0$$

- Heavy-tailed, suitable for modeling financial extremes

Rapidly varying tail

$$\lim_{x\to\infty} \frac{P(R_d>ax)}{P(R_d>x)} = 0,\ a>1$$

- Lighter than any regularly varying tail

…the concern

WHAT IF THE RISK FACTORS ARE NOT ELLIPTICALLY DISTRIBUTED?

Methodology

- Choose a non-elliptical distribution, and aim to evaluate

$$\lim_{\alpha \to 1} \frac{VaR_{\alpha,\, elliptical}(\omega^T X)}{VaR_{\alpha,\, true}(\omega^T X)} = \begin{cases} > 1, \text{overestimation} \\ < 1, \text{underestimation} \end{cases}$$

- Requires an assumption about the tail behavior, i.e., regularly or rapidly varying

Elliptical approximations and

REGULARLY VARYING TAILS

Regularly varying tails

Mixture elliptical distributions

- Multiple market regimes (e.g., normal & stress)
- $C \sim Categorical(p_1, \ldots, p_m)$ determines the regime

$$W =^D \sum_{i=1}^{m} \mu_j 1\{C = j\} + R_d \sum_{i=1}^{m} A_j 1\{C = j\} U^{(d)}$$

- $W|(C = j) =^D \mu_j + R_d A_j U^{(d)}$ is elliptically distributed

Regularly varying tails

Elliptical approximation for mixture elliptical

i) Elliptical $X =^D \mu + R_d A_X U^{(d)}$, $\Sigma_X = A_X A_X^T$,

ii) Mixture elliptical $W =^D \sum_{i=1}^m \mu_j 1\{C = j\} + R_d \sum_{i=1}^m A_j 1\{C = j\} U^{(d)}$, $\Sigma_j = A_j A_j^T$,

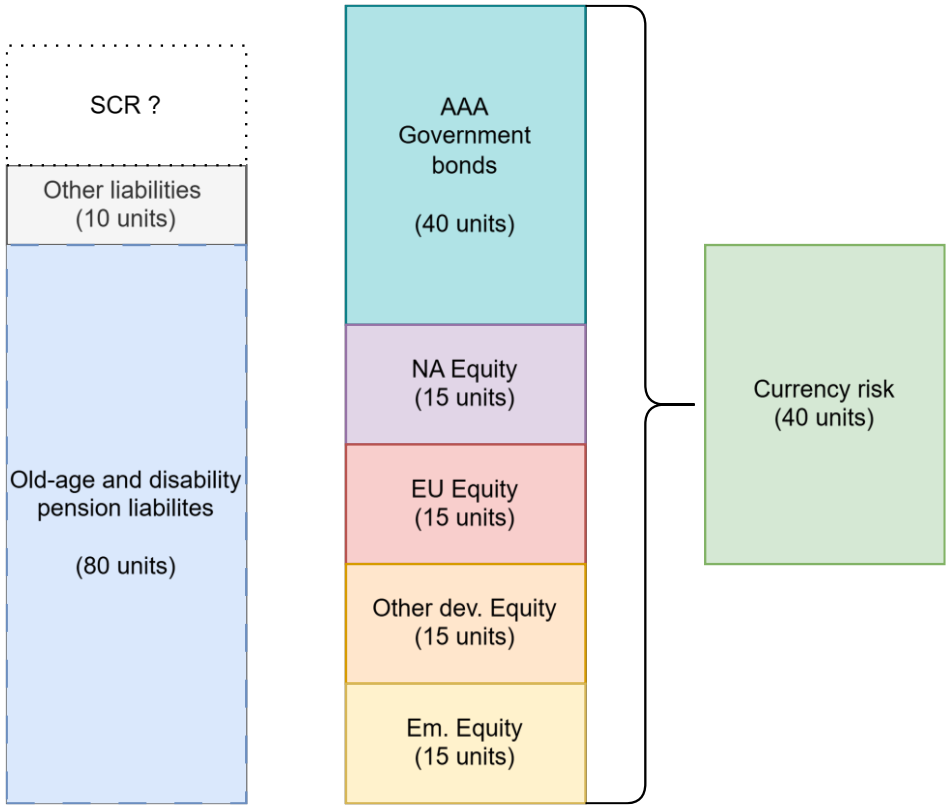
- R_d is regularly varying with index ν, then

$$\lim_{\alpha \to 1} \frac{VaR_\alpha(\omega^T X)}{VaR_\alpha(\omega^T W)} = \frac{\sqrt{\omega^T \Sigma_X \omega}}{\left(\sum_{j=1}^m \sqrt{\omega^T \Sigma_j \omega}^{\nu} \, P(C = j) \right)^{1/\nu}}$$

Regularly varying tails

Example in the Finnish framework

- ***Case study*: Equity returns with two t-distributed correlation regimes**
  - Normal: $\rho_{ij} \in (0.7, 0.8)$
  - Stress: $\rho_{ij} = 0.99$

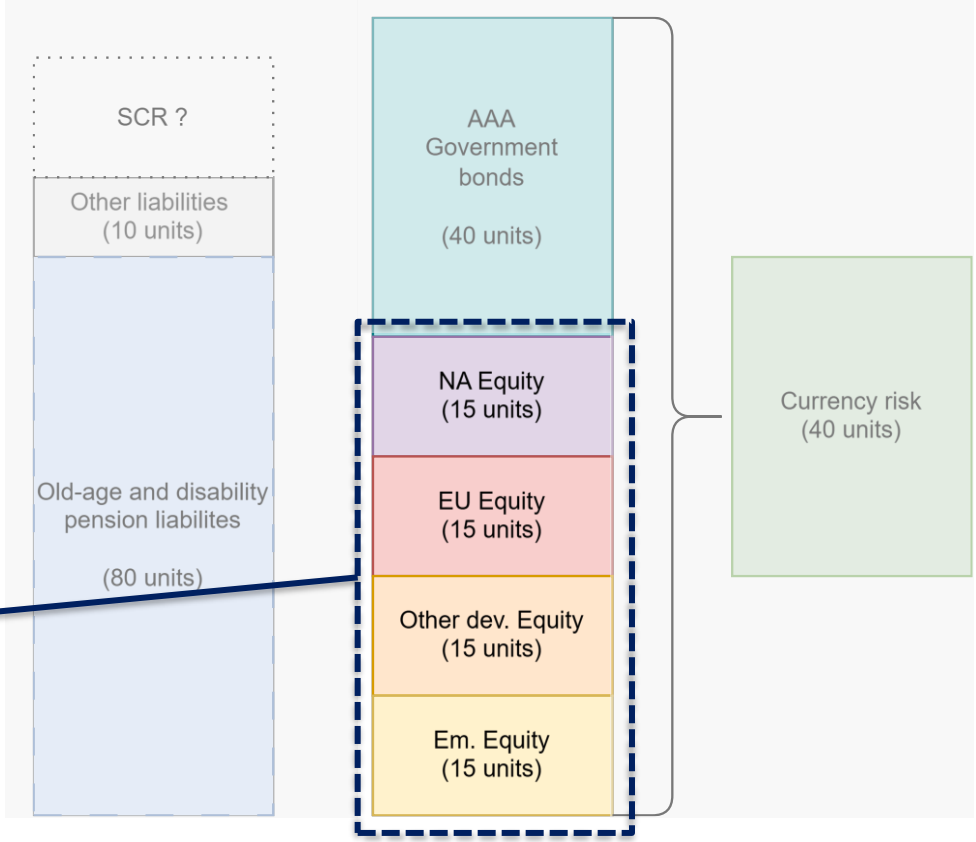


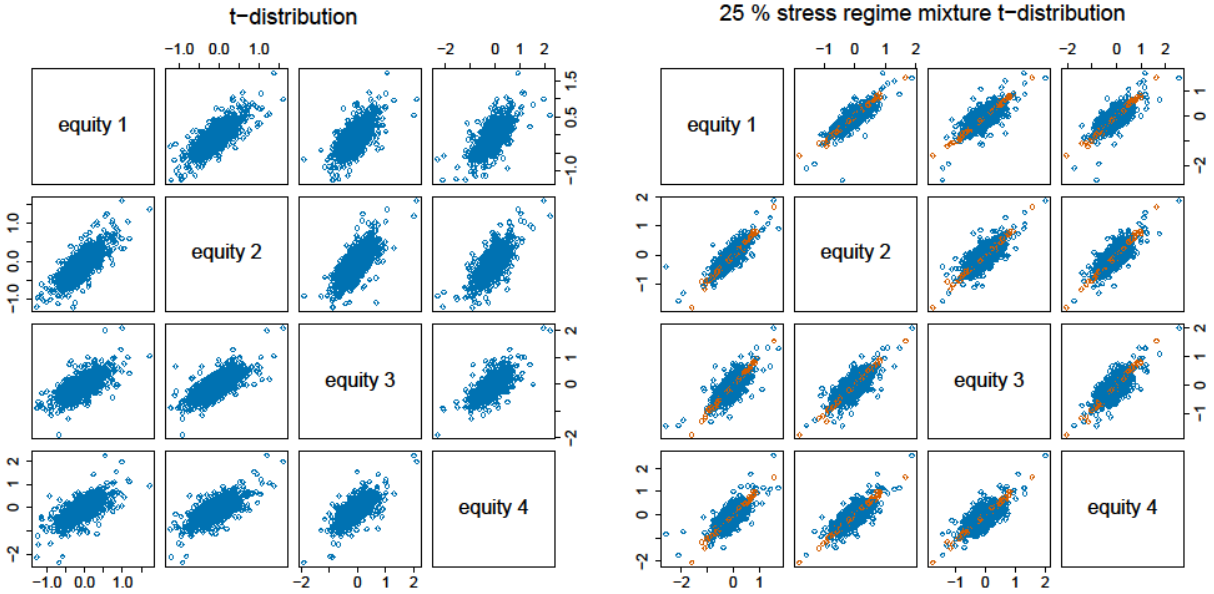
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- Risk factors for the SCR calculation
  - **Equity risks 1, 2, 3 & 4**
  - Bond spread risk
  - Currency risk
  - Underwriting risk
  - ...





Stress regime mixture illustrated

No stress regime vs. 25 % stress regime

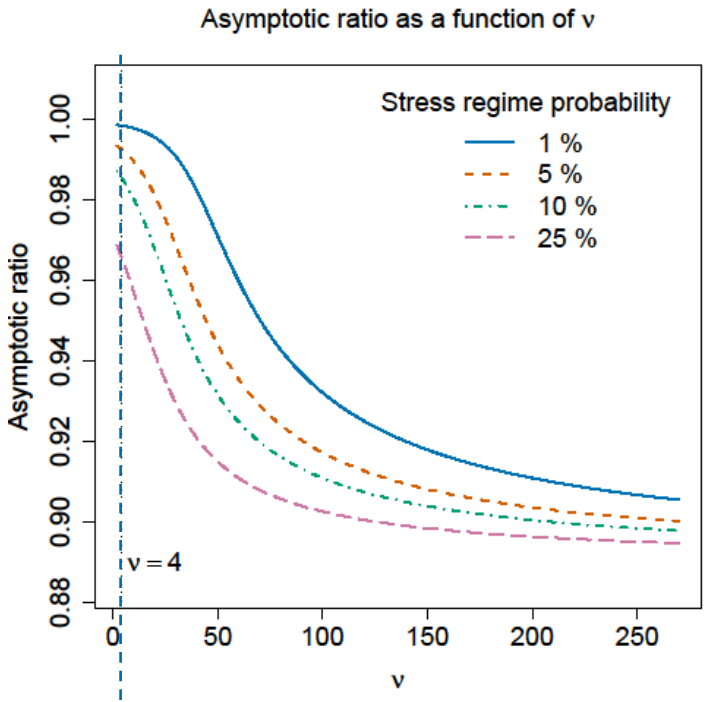
Regularly varying tails, example

Regularly varying tails, example

The Asymptotic ratio

$$\lim_{\alpha \to 1} \frac{VaR_\alpha(\omega^T X)}{VaR_\alpha(\omega^T W)} = \frac{\sqrt{\omega^T \Sigma_X \omega}}{\left(\sum_{j=1}^{m} \sqrt{\omega^T \Sigma_j \omega}^{\nu} P(C = j) \right)^{1/\nu}}$$

- Value depends on tail parameter $\boldsymbol{\nu}$, **stress regime probability**, and weights ω
- Provides a **mathematical explanation** and an estimation method

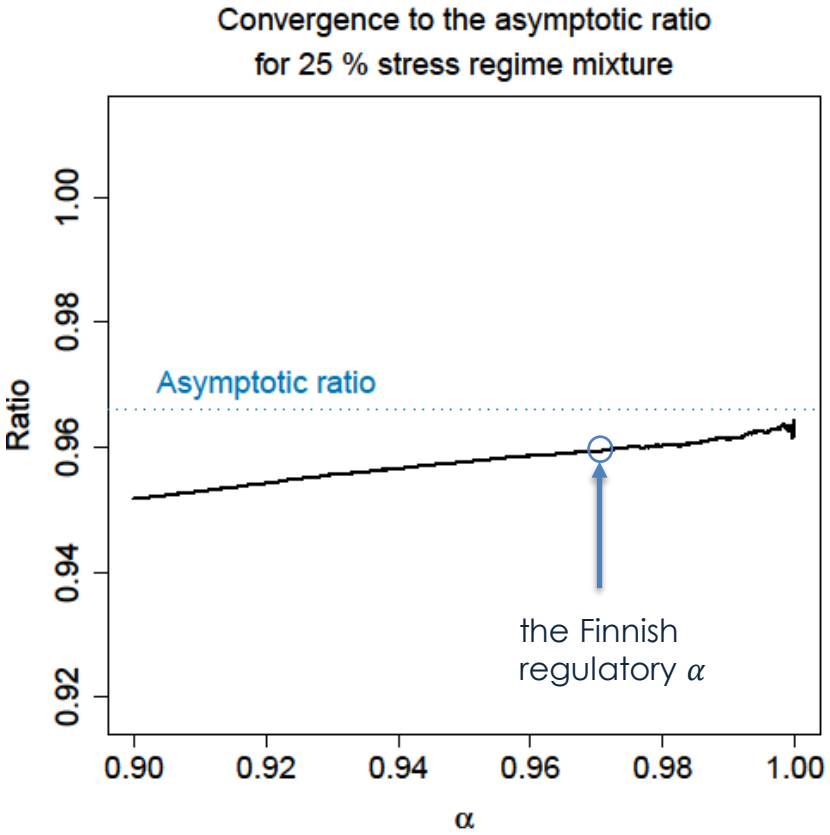


Stress probability	1 %	5 %	10 %	25 %
Asymptotic limit ratio	0.999	0.993	0.986	0.966

Regularly varying tails,example

Does the ratio converge fast enough?

- The speed of convergence depends on multiple factors, for example, ν
- In practice, convergence is verified through simulation
- Fast convergence implies that the ratio is close to its asymptotic value at regulatory α



Regularly varying tails,example

Impact on equity VaR

- The asymptotic ratio helps explain the magnitude of error
- For example, at α = 97 %, 16.722 / 17.431 = 0.959, while the limit is 0.966

α	Stress-regime probability				
	0 %	1 %	5 %	10 %	25 %
95 %	12.790	12.811	12.895	13.013	13.355
97 %	16.722	16.752	16.861	17.007	17.431
99.9 %	55.088	55.162	55.501	55.926	57.179

Regularly varying tails,example

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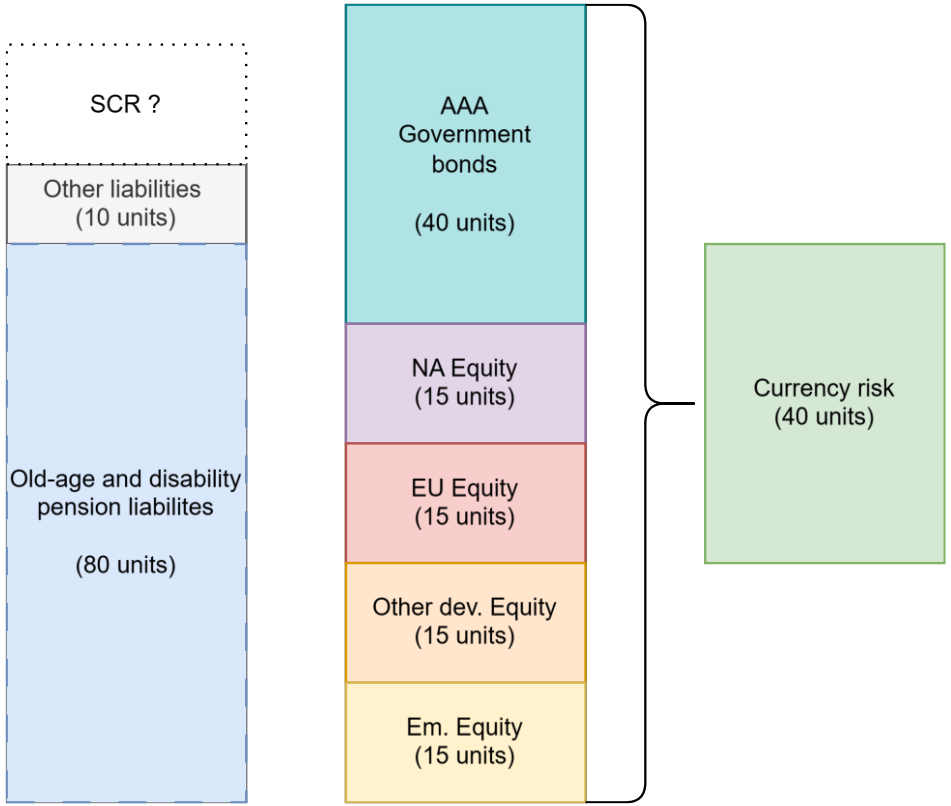
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Regularly varying tails, example

Numerical impact on SCR ($\alpha = 97\%$)

- $\widehat{D}_{equity}(W)$ replaces the regulatory (elliptical) equity risk term by the empirical estimate from the given regime mixture

$$\widehat{SCR}_{estimate} = -\sum_{i=1}^{18} \mu_i + \sqrt{\sum_{i=1}^{18}\sum_{j=1}^{18} \rho_{ij}(V_i - \mu_i)(V_j - \mu_j) + \widehat{D}_{equity}(W)}$$

Stress regime probability	0 %	1 %	5 %	10 %	25 %
$\widehat{SCR}_{estimate}$	13.667	13.707	13.860	14.060	14.638

The other risk factors dilute the overall impact

Conclusion

- The assumption about tail behavior is crucial for the overall accuracy of elliptical approximations
- Elliptical approximations can be more robust than accredited
- For further details, see

  Kuusela, M., Lehtomaa, J., & Pyrylä, A. (2026). On asymptotic behavior of portfolio risks and goodness of elliptical approximations. *Insurance: Mathematics and Economics*, 103275.

Thank you!

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